MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 209
JANUARY 1995, PAGES 71-88

THE SEMIGROUP STABILITY OF THE DIFFERENCE APPROXIMATIONS FOR INITIAL-BOUNDARY VALUE PROBLEMS

LIXIN WU

ABSTRACT. For semidiscrete approximations and one-step fully discretized approximations of the initial-boundary value problem for linear hyperbolic equations with diagonalizable coefficient matrices, we prove that the Kreiss condition is a sufficient condition for the semigroup stability (or l_2 stability). Also, we show that the stability of a fully discretized approximation generated by a locally stable Runge-Kutta method is determined by the stability of the semidiscrete approximation.

1. INTRODUCTION

Consider the following first-order one-dimensional hyperbolic equations:

$$\frac{\partial \mathbf{u}}{\partial t} = A\frac{\partial \mathbf{u}}{\partial x} + B\mathbf{u} + \mathbf{F} \tag{1}$$

in the quarter-plane $\Omega = \{(x, t) \mid x, t \geq 0\}$. Here,

$$\mathbf{u}(x, t) = (u^{(1)}(x, t), \dots, u^{(m)}(x, t))^T$$

and

$$\mathbf{F} = (F^{(1)}(x, t), \dots, F^{(m)}(x, t))^T$$

are vector functions, A and B are $m \times m$ constant matrices. In particular, A is assumed diagonal:

$$A = \begin{pmatrix} A^{\mathrm{I}} & 0 \\ 0 & A^{\mathrm{II}} \end{pmatrix},$$

with

$$A^{\mathrm{I}} = \operatorname{diag}(a_1, a_2, \dots, a_l), \quad a_i < 0, \quad i = 1, \dots, l,$$
$$A^{\mathrm{II}} = \operatorname{diag}(a_{l+1}, a_{l+2}, \dots, a_m), \quad a_i > 0, \quad i = l+1, \dots, m.$$

The solution is uniquely determined [5, 8] if we impose the initial condition

$$\mathbf{u}(x, 0) = \mathbf{f}(x), \qquad x \geq 0, \tag{2}$$

Received by the editor June 15, 1992 and, in revised form, March 5, 1993, June 9, 1993, and October 19, 1993.

1991 *Mathematics Subject Classification*. Primary 65F50, 65Y05; Secondary 65F05.

Key words and phrases. Hyperbolic, semigroup stability, Runge-Kutta methods.

The main part of this work was completed while the author was a student at UCLA. The author wants to thank Professor Kreiss for his support and advice concerning this work, and Professor Osher for many valuable discussions.

and the boundary condition

$$\mathbf{u}^{\mathrm{I}}(0, t) = S\mathbf{u}^{\mathrm{II}}(0, t) + \mathbf{g}(t), \qquad t \geq 0, \tag{3}$$

where $\mathbf{u}^{\mathrm{I}}$ and $\mathbf{u}^{\mathrm{II}}$ are the partitions of $\mathbf{u}$ according to the partition of A, and S is an $l \times (m-l)$ matrix.

A number of theories on the well-posedness of the initial-boundary value problem (1)–(3) have been developed [1, 5, 8, 10, 14]. However, they are not the focus of this paper and will not be addressed here. Instead, we assume the well-posedness of the above initial-boundary value problem (IBV for short) in the so-called semigroup sense.

Assumption 1.1. The IBV problem (1)–(3) is well-posed if

(1) for a dense set of smooth data there is a smooth solution;
(2) the solution of the homogeneous equations ($\mathbf{F} = 0$) with homogeneous boundary condition ($\mathbf{g} = 0$) satisfies

$$\|\mathbf{u}(\cdot, t)\| \leq Ke^{\eta_0 t}\|\mathbf{u}(\cdot, 0)\|, \tag{4}$$

where η_0 and K are universal constants.

Note that $\|\cdot\|$ is the usual L^2 norm in the half-space. For the solution of inhomogeneous equations, one can obtain estimates by Duhamel's principle. The above definition is the most natural way to define stability for hyperbolic Cauchy problems.

Now we start considering the numerical solution of the IBV problem (1)–(3) by finite difference methods. We introduce a mesh of size $h = \Delta x > 0$ and $k = \Delta t > 0$ in the quarter-plane and, using the notation $\mathbf{u}_\nu^n \approx \mathbf{u}(\nu h, nk)$, approximate the equations by a consistent one-step scheme of the form

$$\begin{aligned} &\mathbf{u}_\nu^{n+1} = Q_0\mathbf{u}_\nu^n + k\tilde{\mathbf{F}}_\nu^n, \qquad \nu = 1, 2, \dots, \\ &Q_0 = \sum_{j=-r}^{p} A_j E^j, \qquad E\mathbf{u}_\nu = \mathbf{u}_{\nu+1}, \end{aligned} \tag{5}$$

where the $m \times m$ matrices A_j are polynomials in A and kB, and the m-vector $\tilde{\mathbf{F}}_\nu(t)$ is a smooth function of $\mathbf{F}$ and its derivatives. The initial condition for (5) is

$$\mathbf{u}_\nu^0 = \mathbf{f}_\nu := \mathbf{f}(h\nu), \qquad \nu = 1, 2, \dots. \tag{6}$$

In addition to the physical boundary condition (3), there are usually numerical boundary conditions required in (5). The boundary conditions, physical or numerical, can be put together as

$$\mathbf{u}_\mu^n = \sum_{j=1}^{q} C_{j\mu}E^j\mathbf{u}_\mu^n + \mathbf{g}_\mu^n, \qquad \mu = -r+1, -r+2, \dots, 0, \tag{7}$$

where $C_{j\mu}$ are constant matrices.

The *methods of lines*, as the major techniques to generate high-order schemes, deserve particular attention. These techniques simply couple the spatial discretization with numerical ODE schemes for time stepping. Let $\mathbf{u}_\nu(t) \approx \mathbf{u}(\nu h, t)$; we approximate (1) by a consistent semidiscrete scheme of the form

$$\begin{aligned} \frac{d\mathbf{u}_\nu(t)}{dt} &= Q\mathbf{u}_\nu(t) + \mathbf{F}_\nu(t), \quad \nu = 1, 2, \dots, \\ Q &= \sum_{j=-r}^{p} A_j E^j, \end{aligned} \tag{8}$$

where A_j are linear functions of A and B. The initial condition follows naturally from (2), i.e,

$$\mathbf{u}_\nu(0) = \mathbf{f}_\nu := \mathbf{f}(\nu h), \quad \nu = 1, 2, \dots. \tag{9}$$

Boundary conditions, both physical and artificial, are implemented as

$$\mathbf{u}_\mu(t) = \sum_{j=1}^{q} C_{j\mu} E^j \mathbf{u}_\mu(t) + \mathbf{g}_\mu(t), \quad \mu = -r+1, -r+2, \dots, 0. \tag{10}$$

This system of ordinary differential equations is then solved by standard numerical methods for ODEs. Inevitably, the well-posedness of the ODE system must be dealt with before one proceeds to its numerical solution.

For the fully discretized and semidiscretized problems, their formal solutions always exist. Thus, stability is the only concern. In this paper, we will consider the semigroup stability for the discretized problems. That is, putting $\mathbf{F} = 0$ and $\mathbf{g} = 0$, we discuss the conditions under which (4) will hold, with the L^2 norm being replaced by its discrete version.

In the subsequent sections we will show that for semidiscrete approximations and one-step fully discretized approximations to the IBV problem (1)–(3), Kreiss' condition, the sufficient and necessary condition for GKS stability [4], is a sufficient condition for the semigroup stability (or l_2 stability). This result disperses the long-standing mist over the relation between the semigroup stability and the GKS stability for the discretized problems. For semidiscrete approximations, this paper offers a satisfactory answer to the quest for a theory of semigroup stability [3]. For fully discretized approximations, our results supersede and generalize the other two classical theories by Kreiss [6, 7] and Osher [13].

Before we finish this section, we introduce some notations. We denote the solution space of the IBV problem (5)–(7) and (8)–(10) by $l^{2,m}(1, \infty)$, which is defined by

$$l^{2,m}(-M, N) = \{\mathbf{u} = \{\mathbf{u}_j\}_{-M}^{N}, \mathbf{u}_j \in \mathbf{C}^m \mid \|\mathbf{u}\|_{-M,N} < \infty\}.$$

The norm comes from the associated inner product

$$(\mathbf{u}, \mathbf{v})_{-M,N} = \sum_{-M}^{N} \mathbf{u}_j^* \mathbf{v}_j h.$$

Thus,

$$\|\mathbf{u}\|_{-M,N}^2 = (\mathbf{u}, \mathbf{u})_{-M,N}.$$

For the sake of convenience, the indices of the norm and inner product of $l^{2,m}(1, \infty)$ will be omitted. We write $l^2(-M, N)$ for $l^{2,1}(-M, N)$.

2. Semidiscrete approximations

2.1. **Prerequisites.** For (8)–(10) to be stable in the semigroup sense, it is necessary that (8) be semigroup stable for the Cauchy problem. In other words, we will only be interested in those spatial discretizations which satisfy

Assumption 2.1. The operator Q for the Cauchy problem of the semidiscrete equations (8) is semibounded, i.e, there exists a real constant η_0 such that

$$(\mathbf{u}, Q\mathbf{u})_{-\infty,\infty} + (Q\mathbf{u}, \mathbf{u})_{-\infty,\infty} \le 2\eta_0(\mathbf{u}, \mathbf{u})_{-\infty,\infty}.$$

An immediate consequence of semiboundedness of Q is that, when $\mathbf{F} = 0$, the solution of (8) satisfies

$$\|\mathbf{u}(t)\|_{-\infty,\infty} \le Ce^{\eta_0 t}\|\mathbf{u}(0)\|_{-\infty,\infty}.$$

Our theory will build upon the classical GKS stability theory [15]. Hence, a brief description of the main results of the GKS theory will be given below. The GKS stability is defined by

Definition 2.1. The discrete problem (8)–(10) is stable if for $\eta > \eta_0$ the solution of the problem with homogeneous initial value $(\mathbf{f} = 0)$ satisfies

$$\begin{aligned}(11)\qquad &\int_0^\infty (|\mathbf{u}(0, t)|_{\mathscr{B}}^2 + (\eta - \eta_0)\|\mathbf{u}(\cdot, t)\|^2)e^{-2\eta t}dt \\ &\le K\int_0^\infty \left(|\mathbf{g}|_{\mathscr{B}}^2 + \frac{1}{\eta - \eta_0}\|\mathbf{F}(\cdot, t)\|^2\right)e^{-2\eta t}dt,\end{aligned}$$

where η_0, K are universal constants.

The terms with index $\mathscr{B}$ are boundary norms defined by

$$(12)\qquad |\mathbf{u}|_{\mathscr{B}} = \sum_{j=-r+1}^{0} |\mathbf{u}_j|, \quad |\mathbf{u}_j| = \sum_{i=1}^{m} |u_j^{(i)}|.$$

The necessary and sufficient condition for GKS stability is determined by an eigenvalue problem, which is obtained by taking the Laplace transform of the homogenized equations of (8)–(10):

$$\begin{aligned}(13)\qquad &s\hat{\mathbf{u}}_\nu = Q\hat{\mathbf{u}}_\nu, \qquad \operatorname{Re}(s) \ge 0, \quad \nu = 1, 2, \dots, \\ &\hat{\mathbf{u}}_\mu = \sum_{j=1}^{q} C_{\mu j}\hat{\mathbf{u}}_{\mu+j}, \quad \mu = -r+1, -r+2, \dots, 0,\end{aligned}$$

where

$$\hat{\mathbf{u}}(x, s) = \int_0^\infty e^{-st}\mathbf{u}(x, t)dt, \quad \operatorname{Re}(s) \ge 0.$$

The eigenvalues and generalized eigenvalues of (13) are defined below.

Definition 2.2. Let $B = 0$ and $\operatorname{Re}(s) \ge 0$. Then s is called an eigenvalue if

1. there exists a nontrivial solution $\hat{\mathbf{u}}$ to (13);
2. $\|\hat{\mathbf{u}}\| < \infty$ for $\operatorname{Re}(s) > 0$.

We call s a generalized eigenvalue if it satisfies condition 1 and

2′. $\|\hat{\mathbf{u}}\| = \infty$. Furthermore, $\hat{\mathbf{u}}_j(s) = \lim_{\theta \to s} \hat{\mathbf{u}}_j(\theta)$ with $\mathrm{Re}(\theta) > 0$, and $\hat{\mathbf{u}}(\theta)$ satisfies $(\theta I - Q)\hat{\mathbf{u}}(\theta) = 0$.

We can now state

Theorem 2.1 (Strikwerda [15]). *The approximation* (8)–(10) *is GKS stable if and only if* (13) *has no eigenvalue nor generalized eigenvalue on the half-plane* $\mathrm{Re}(s) \geq 0$.

The eigenvalue condition in the above theorem is usually referred to as the Kreiss' condition. Sometimes, it is more convenient to use the following interpretation of the Kreiss' condition [4, 15].

Lemma 2.1. *For the semidiscrete approximation* (8)–(10), *Kreiss' condition is equivalent to: when* $\mathbf{F} = 0$ *and* $\mathbf{f} = 0$, *there holds*

$$|\hat{\mathbf{u}}_j| \leq K_j |\hat{\mathbf{g}}|_{\mathscr{B}}, \qquad j \geq -r+1, \tag{14}$$

where K_j *is a constant depending on* j *only.*

There is an additional assumption in the GKS theory which has to be included in this paper as well. Note that we have found no semidiscrete approximation which violates this assumption.

Assumption 2.2. The basic scheme (8) is either dissipative or nondissipative, i.e, the roots of the characteristic equation

$$\det |\, sI - \hat{Q}(i\xi)\,| = 0, \quad \text{where} \quad \hat{Q}(i\xi) = \sum_{j=-r}^{p} A_j e^{ij\xi}, \tag{15}$$

satisfy either

$$\mathrm{Re}(s) < 0, \quad 0 < |\xi| \leq \pi,$$

or

$$\mathrm{Re}(s) = 0, \quad |\xi| \leq \pi.$$

Finally, in this section we claim that it suffices to discuss the stability issues for a scalar problem. This is based on the fact that semigroup stability is stable against lower-order perturbations. The exact meaning of this statement is illustrated in the following

Lemma 2.2 [11]. *Suppose the solution of the infinite system of ordinary differential equations*

$$\frac{d\mathbf{u}}{dt} = Q\mathbf{u}$$

satisfies the energy estimate

$$\|\mathbf{u}(t)\| \leq K e^{\eta_0 t} \|\mathbf{u}(0)\|.$$

Let H *be any bounded linear operator with* $\|H\| \leq \beta$. *Then the solution of the perturbed system*

$$\frac{d\mathbf{w}}{dt} = (Q + H)\mathbf{w}$$

satisfies

$$\|\mathbf{w}(t)\| \leq K e^{\gamma t} \|\mathbf{w}(\cdot, 0)\|, \quad \gamma = \eta_0 + K\beta.$$

The above lemma tells us that lower-order terms will not affect the stability property of an ODE system. Thus, they can be neglected for the purpose of stability discussions. Once all lower-order terms are ignored, the equations become decoupled (except in the boundary conditions). Under this circumstance, all our discussions and assertions for a single scalar equation are formally the same as those for a system of equations. For the sake of simplicity, we will proceed with a single equation in the subsequent sections, and indicate results for a system of equations accordingly.

2.2. **Kreiss' condition and semigroup stability.** As we have previously explained, we only need to consider the scalar problem

$$\begin{cases} \dfrac{du_j(t)}{dt} = Qu_j(t), & j = 1, 2, \dots, \\ u_j(0) = f_j, \\ \mathscr{B}u_0(t) = 0, \end{cases} \tag{16}$$

where

$$Q = \frac{1}{h}\sum_{j=-r}^{p} a_\mu E^j, \quad \text{with } a_{-r} \neq 0, a_p \neq 0,$$

is the difference approximation of $a\frac{\partial}{\partial x}$. The operator $\mathscr{B}$ represents a set of boundary conditions of the form

$$u_\mu(t) = \sum_{j=1}^{q} \beta_{j\mu} u_{\mu+j}, \qquad -r+1 \leq \mu \leq 0,$$

which make Q well defined in $l^2(1, \infty)$.

Our fundamental technique here is to construct a set of special boundary conditions for our semidiscrete equation such that its solution at every line $x = x_j$ can be bounded in terms of the initial values:

$$\int_0^\infty e^{-2\eta_0 t}|u_j(t)|^2 dt \leq c_j \|u(0)\|^2, \quad j = 1, 2, \dots,$$

where c_j depends on j only. We then subtract this auxiliary problem from the original one (8)–(10). In this way, the original problem with inhomogeneous initial data is reduced to the problem with inhomogeneous boundary conditions but homogeneous initial data. Then, Lemma 2.1 and the energy estimates will lead to our results. This approach had been suggested by Kreiss [9] and Gustafsson [3]. Kreiss constructed such special boundary conditions for a particular problem [9].

Recalling Assumption 2.1, we know that for any $u \in l^2(-\infty, \infty)$,

$$\operatorname{Re}(u, Qu)_{-\infty,\infty} \leq \eta_0(u, u)_{-\infty,\infty}.$$

From this, we have

Theorem 2.2. *There exists a boundary operator $\mathscr{B}_0$ such that for all $u \in l^2(1, \infty)$ satisfying $\mathscr{B}_0 u_0(t) = 0$, the following inequality holds:*

$$\text{Re}(u, Qu) \le \eta_0(u, u) - c \sum_{j=-r+1}^{r} |u_j|^2, \tag{17}$$

where $c > 0$ is a constant.

Proof. By writing the boundary terms, which are not yet determined, as a single vector $u^b \in l^2(-\infty, \infty)$,

$$(u^b(t))_j = \begin{cases} u_j(t) & \text{for } -r+1 \le j \le 0, \\ 0 & \text{otherwise}, \end{cases}$$

and introducing an injection $I: l^2(1, \infty) \to l^2(-\infty, \infty)$:

$$(Iu)_j = \begin{cases} u_j, & j = 1, 2, \dots, \\ 0, & j \le 0, \end{cases}$$

we can rewrite the inner products in $l^2(1, \infty)$ by those in $l^2(-\infty, \infty)$:

$$\begin{aligned}(u, Qu) &= (Iu, Q(Iu + u^b))_{-\infty,\infty} \\ &= (Iu, QIu)_{-\infty,\infty} + (Iu, Qu^b)_{-\infty,\infty}.\end{aligned}$$

Then we have, after taking the real part of each term,

$$\begin{aligned}\text{Re}(u, Qu) &\le \eta_0(u, u) + \text{Re}(Iu, Qu^b)_{-\infty,\infty} \\ &= \eta_0(u, u) + \text{Re}(u, Qu^b)_{1,r} \\ &= \eta_0(u, u) + \text{Re}\{U^* Q_1 U^b\},\end{aligned}$$

where

$$\begin{aligned}U &= (u_1, u_2, \dots, u_r)^T, \\ U^b &= (u_{-r+1}, u_{-r+2}, \dots, u_0)^T,\end{aligned}$$

and Q_1 is the $r \times r$ nonsingular triangular matrix

$$Q_1 = \begin{pmatrix} a_{-r} & a_{-r+1} & a_{-r+2} & \cdots & a_{-2} & a_{-1} \\ 0 & a_{-r} & a_{-r+1} & a_{-r+2} & \cdots & a_{-2} \\ 0 & 0 & a_{-r} & a_{-r+1} & \cdots & a_{-3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & a_{-r} \end{pmatrix}.$$

Hence, if we choose

$$U^b = -Q_1^{-1} U, \tag{18}$$

then we have

$$\text{Re}(u, Qu) \le \eta_0(u, u) - \sum_{j=1}^{r} |u_j|^2,$$

or

$$\operatorname{Re}(u, Qu) \leq \eta_0(u, u) - \sigma_1^2(Q_1) \sum_{j=-r+1}^{0} |u_j|^2,$$

where $\sigma_1(Q_1)$ is the smallest singular value of Q_1. Combining the above two inequalities, we obtain (17). The boundary operator corresponding to (18) is the required operator $\mathscr{B}_0$. □

With the special boundary operator $\mathscr{B}_0$, the solution of (16) satisfies

$$e^{-2\eta_0 t}\|u(t)\|^2 + c\int_0^t \sum_{j=-r+1}^{r} e^{-2\eta_0\tau}|u_j|^2 d\tau = \|u(0)\|^2. \tag{19}$$

If we introduce the notation

$$\langle u_j, u_j\rangle = \int_0^\infty e^{-2\eta_0 t}|u_j(t)|^2 dt,$$

then (19) yields

$$\langle u_j, u_j\rangle \leq \text{const}\,\|u(0)\|^2, \quad j = -r+1, \dots, r.$$

Next we will show that with this special boundary operator $\mathscr{B}_0$, we can obtain estimates for all u_j. Considering u_j, $j = 1, \dots, r$, as known, we treat u_j, $j = r+1, \dots, \infty$, as the solution of the following system:

$$\begin{cases} \dfrac{du_j}{dt}(t) = Qu_j(t), & j = r+1, r+2, \dots \\ u_j(0) = f_j, & \\ u_\mu, \quad 1 \leq \mu \leq r, \text{ known}, & \end{cases}$$

and try to estimate u_j, $j = r+1, \dots, 2r$. For this purpose, we split $u_j(t)$, $j \geq r+1$, into

$$u_j(t) = v_j(t) + (u_j(t) - v_j(t)), \quad j = r+1, r+2, \dots,$$

where $v_j(t)$, $j \geq r+1$, satisfies

$$\begin{cases} \dfrac{dv_j}{dt}(t) = Qv_j(t), & j = r+1, r+2, \dots, \\ v_j(0) = f_j, & \\ \mathscr{B}_0 v_r = 0, & \end{cases}$$

and $w_j(t) := u_j(t) - v_j(t)$, $j \geq r+1$, satisfies

$$\begin{cases} \dfrac{dw_j}{dt}(t) = Qw_j(t), & j = r+1, r+2, \dots, \\ w_j(0) = 0, & \\ w_\mu = u_\mu - v_\mu, \quad 1 \leq \mu \leq r. & \end{cases} \tag{20}$$

From Theorem 2.2 and (19), we have

$$\langle v_j, v_j\rangle \leq \|u(0)\|^2_{r+1,\infty} \leq \|u(0)\|^2, \quad j = r+1, \dots, 2r.$$

Thus, we only need to estimate w_j, $j = r+1, \dots, 2r$. Taking the Laplace transform of (20) for $s = \eta + i\xi$ with $\eta \geq \eta_0$, we have

$$\begin{cases} s\hat{w}_j = Q\hat{w}_j, & j = r+1, r+2, \dots, \\ \hat{w}_\mu = \hat{u}_\mu - \hat{v}_\mu, & \mu = 1, \dots, r. \end{cases} \tag{21}$$

Its corresponding eigenvalue problem is

$$s\hat{w}_j = Q\hat{w}_j, \quad j = r+1, r+2, \dots, \tag{22}$$

$$\hat{w}_\mu = 0, \quad \mu = 1, \dots, r. \tag{23}$$

The characteristic equation of (22) is

$$sh = \sum_{j=-r}^{p} a_j \kappa^j. \tag{24}$$

Its roots are continuous functions of $\tilde{s} := sh$. Let $\kappa_\alpha = \kappa_\alpha(\tilde{s})$, $1 \leq \alpha \leq l$, be those roots of (24) lying inside the unit circle when $\mathrm{Re}(\tilde{s}) > 0$, and let $m_\alpha = m_\alpha(\tilde{s})$ denote the multiplicity of $\kappa_\alpha(\tilde{s})$. The general solution of (22) in $l^2(1, \infty)$ is given by

$$\hat{w}_j = \sum_{\alpha=1}^{l} \sum_{\beta=0}^{m_\alpha - 1} c_{\alpha\beta} P_{\alpha\beta}(j) \kappa_\alpha^j,$$

where $P_{\alpha\beta}(j)$ are arbitrary polynomials in j with degree exactly equal to β, and $c_{\alpha\beta}$ are parameters determined by the boundary conditions (23), which now read

$$\sum_{\alpha=1}^{l} \sum_{\beta=0}^{m_\alpha - 1} c_{\alpha\beta} P_{\alpha\beta}(\mu) \kappa_\alpha^\mu = 0, \quad \mu = 1, \dots, r. \tag{25}$$

Note that the number of these roots is equal to the number of boundary conditions, i.e.,

$$\sum_{\alpha=1}^{m} m_\alpha = r.$$

Thus, the number of the parameters $c_{\alpha\beta}$ is exactly equal to the number of equations in (25). We will show that the only solution of (22), (23) is the trivial solution. This will be done using a technique introduced by Goldberg and Tadmor [2].

Lemma 2.3. *The eigenvalue problem* (22), (23) *satisfies the Kreiss condition.*

Proof. We make a special selection of $P_{\alpha\beta}$:

$$P_{\alpha\beta}(\mu) = \kappa_\alpha^{-1-\beta} \beta! \binom{\mu - 1}{\beta}.$$

Then (25) becomes

$$\sum_{\alpha=1}^{l} \sum_{\beta=0}^{m_\alpha - 1} \beta! \binom{\mu - 1}{\beta} \kappa_\alpha^{\mu-1-\beta} c_{\alpha\beta} = 0, \quad \mu = 1, \dots, r,$$

that is, with $\nu = \mu - 1$,

$$\sum_{\alpha=1}^{l} \sum_{\beta=0}^{m_\alpha - 1} \frac{\partial^\beta \kappa_\alpha^\nu}{\partial \kappa_\alpha^\beta} c_{\alpha\beta} = 0, \quad \nu = 0, 1, \dots, r-1. \tag{26}$$

The coefficient matrix of the above system is

$$J = [B(\kappa_1, m_1), \dots, B(\kappa_l, m_l)],$$

where, for $\alpha = 1, \dots, l$,

$$B(k_\alpha, m_\alpha) = \left[\begin{pmatrix} \kappa^{r-1} \\ \kappa^{r-2} \\ \vdots \\ 1 \end{pmatrix}, \frac{\partial}{\partial \kappa} \begin{pmatrix} \kappa^{r-1} \\ \kappa^{r-2} \\ \vdots \\ 1 \end{pmatrix}, \dots, \frac{\partial^{m_\alpha - 1}}{\partial \kappa^{m_\alpha - 1}} \begin{pmatrix} \kappa^{r-1} \\ \kappa^{r-2} \\ \vdots \\ 1 \end{pmatrix} \right]_{\kappa = \kappa_\alpha}.$$

Let $\mathbf{c} = (c_1, c_2, \dots, c_r)^T$ be a vector such that

$$J^T \mathbf{c} = 0.$$

That is,

$$\left\{ \sum_{\nu=0}^{r-1} c_\nu \frac{\partial^\beta}{\partial \kappa^\beta} \kappa^\nu \right\}_{\kappa = \kappa_\alpha} = 0, \quad 0 \le \beta \le m_\alpha - 1, \qquad 1 \le \alpha \le l,$$

or

$$\frac{\partial^\beta}{\partial \kappa^\beta} \left\{ \sum_{\nu=0}^{r-1} c_\nu \kappa^\nu \right\}_{\kappa = \kappa_\alpha} = 0, \quad 0 \le \beta \le m_\alpha - 1, \qquad 1 \le \alpha \le l.$$

From the above relations we conclude that the polynomial

$$P(\kappa) := \sum_{\mu=0}^{r-1} c_\mu \kappa^\mu$$

has r roots. Since $P(\kappa)$ is of degree $r-1$, this means $P(\kappa) \equiv 0$. So, we have $c_\nu = 0$, $\nu = 1, \dots, r-1$, which implies that J is nonsingular. Thus the system (26) has no nontrivial solution and the lemma is proved. □

According to Lemma 2.1, we can estimate the solution of (21) in terms of the boundary data,

$$|\hat{w}_j| \le \text{const} \sum_{\mu=1}^{r} |\hat{w}_\mu|, \quad j = r+1, r+2, \dots.$$

Hence, for $j = r+1, \dots, 2r$,

$$|\hat{u}_j| \le |\hat{w}_j| + |\hat{v}_j| \le \text{const} \sum_{i=1}^{r} (|\hat{w}_i| + |\hat{v}_j|).$$

With Parseval's equality, these inequalities lead to the desired estimates

$$\langle u_j, u_j \rangle \le c_j \|u(0)\|^2, \quad j = r+1, \dots, 2r.$$

The approach used to derive the estimates for $\langle u_j, u_j\rangle$, $j = r+1, \dots, 2r$, can be used inductively to derive the estimates for all $\langle u_j, u_j\rangle$. Hence, we arrive at

Lemma 2.4. *For scheme* (16) *with special boundary operator* $\mathscr{B}_0$, *we have*

$$\langle u_j, u_j\rangle \le c_j \|u(0)\|^2, \quad j \ge 1,$$

where c_j *depends on* j *only.*

Now we can formulate our major result about the semigroup stability of (16) in terms of the eigenvalue problem:

$$\begin{cases} s\hat{u}_j = Q\hat{u}_j, & j \ge 1, \\ \mathscr{B}\hat{u}_0 = 0. \end{cases} \tag{27}$$

Theorem 2.3. *If* Q *is semibounded for the Cauchy problem and the eigenvalue problem* (27) *satisfies the Kreiss condition, then the IBV problem* (16) *is stable in the semigroup sense.*

Proof. Let v denote the solution of

$$\begin{cases} \dfrac{dv_j(t)}{dt} = Qv_j(t), & j = 1, 2, \dots, \\ v_j(0) = f_j, \\ \mathscr{B}_0 v_0(t) = 0. \end{cases} \tag{28}$$

From Lemma 2.4 we have

$$\langle v_j, v_j\rangle \le c_j \|u(0)\|^2, \quad j \ge 1.$$

Let $w := u - v$. It satisfies

$$\begin{cases} \dfrac{dw_j(t)}{dt} = Qw_j(t), & j = 1, 2, \dots, \\ w_j(0) = 0, \\ \mathscr{B}w_0(t) = -\mathscr{B}v_0(t). \end{cases} \tag{29}$$

Taking the Laplace transform of the above equation, we have

$$\begin{cases} s\hat{w}_j = Q\hat{w}_j, & j \ge 1, \\ \mathscr{B}\hat{w}_0 = -\mathscr{B}\hat{v}_0. \end{cases} \tag{30}$$

The Kreiss condition implies that the solution at the boundary can be estimated in terms of the data. Let $p = \max\{q, r\}$. Then there is a constant c_p such that

$$|\hat{w}|^2_{\mathscr{B}_p} \equiv \sum_{j=-r+1}^{p} |\hat{w}_j|^2 \le c_p \sum_{j=-r+1}^{p} |\hat{v}_j|^2.$$

Hence,

$$\int_0^\infty e^{-2\eta_0 t} |w(t)|^2_{\mathscr{B}_p} dt \le c_p \int_0^\infty e^{-2\eta_0 t} \sum_{j=-r+1}^{p} |v_j(t)|^2 dt \le \text{const}\, \|u(0)\|^2.$$

Because

$$\frac{d(w, w)}{dt} = 2(w, Qw) \le 2\eta_0 (w, w) + \text{const}\, |w|^2_{\mathscr{B}_p},$$

we have

$$\|w(t)\|^2 \le \text{const}\, e^{2\eta_0 t}\|u(0)\|^2.$$

This leads immediately to

$$\|u(t)\| \le \|v(t)\| + \|w(t)\| \le \text{const}\, e^{\eta_0 t}\|u(0)\|.$$

The result is thus proved. □

The corresponding result for a system of equations is

Theorem 2.4. *If Q is semibounded for the Cauchy problem, then the IBV problem* (8)–(10) *is stable in the semigroup sense if the Kreiss condition is satisfied.*

3. One-step fully discretized approximations

3.1. **Prerequisites.** Consider the stability of the fully discretized problem (5)–(7). It is natural to require that (5) be stable in the semigroup sense for the pure Cauchy problem, i.e., we need

Assumption 3.1. For the one-step scheme (5), there is a CFL number $\lambda_0 > 0$ such that when $0 \le \lambda \le \lambda_0$, the solution of the corresponding Cauchy problem with $\tilde{\mathbf{F}} = 0$ satisfies

$$(\mathbf{u}^n, \mathbf{u}^n)_{-\infty,\infty} \le e^{2\eta_0 t}(\mathbf{u}^0, \mathbf{u}^0)_{-\infty,\infty} \tag{31}$$

for some real number η_0.

Remark. For multistep schemes we usually do not have (31).

We will need some results of the GKS theory. We begin with an eigenvalue problem which is obtained by taking the Laplace transform of the homogenized equations of (5)–(7):

$$z\hat{\mathbf{u}}_\nu = Q_0\hat{\mathbf{u}}_\nu, \qquad z = e^{sk}, \quad \nu = 1, 2, \dots, \tag{32}$$

$$\hat{\mathbf{u}}_\mu = \sum_{j=1}^{q}\beta_{j\mu}\hat{\mathbf{u}}_{\mu+j}, \quad \mu = -r+1, -r+2, \dots, 0. \tag{33}$$

The eigenvalues and generalized eigenvalues are defined by

Definition 3.1. Let $B = 0$ and $|z| \ge 1$. Then z is called an eigenvalue if

1. there is a nontrivial solution $\hat{\mathbf{u}}$ to (32), (33);
2. $\|\hat{\mathbf{u}}\| < \infty$ for $|z| > 1$.

We call z a generalized eigenvalue if it satisfies condition 1 and

2′. $\|\hat{\mathbf{u}}\| = \infty$. Furthermore, $\hat{\mathbf{u}}_\nu(z) = \lim_{\theta\to z, |\theta|>1}\hat{\mathbf{u}}_\nu(\theta)$, and $\mathbf{u}_\nu(\theta)$ satisfies $(\theta I - Q_0)\hat{\mathbf{u}}(\theta) = 0$.

With these definitions, we can state the condition for GKS stability [4] of (5)–(7). Note that the temporal integration in Definition 2.1 now implies a summation:

$$\int_0^\infty w(t)dt = \sum_{n=0}^{\infty} w(nk)k.$$

Theorem 3.1 (Gustafsson, Kreiss and Sundström). *The approximation* (5)–(7) *is GKS stable if and only if the eigenvalue problem* (32), (33) *has no eigenvalues nor generalized eigenvalues for* $|z| \geq 1$.

With boundary norms (12), the above theorem can be restated as

Lemma 3.1 (Gustafsson, Kreiss and Sundström). *For discrete approximations* (5)–(7), *Kreiss' condition is equivalent to: when* $\mathbf{F} = 0$, $\mathbf{f} = 0$, *then*

$$|\hat{\mathbf{u}}_j| \leq K_j |\hat{\mathbf{g}}|_{\mathscr{B}}, \qquad j \geq -r+1, \tag{34}$$

where K_j *is a constant depending on* j *only.*

Similarly to the semidiscrete case, the following additional assumption is needed. Again, among the schemes in use, we have found no violation of this assumption.

Assumption 3.2. The basic scheme (5) is either dissipative or nondissipative, i.e, the roots of the characteristic equation

$$\det \left| zI - \sum_{\sigma=0}^{s} \hat{Q}_\sigma(i\xi) z^{-\sigma k} \right| = 0, \quad \text{where } \hat{Q}_\sigma(i\xi) = \sum_{j=-r}^{p} A_{j\sigma} e^{ij\xi}, \tag{35}$$

satisfy either

$$|z(\xi)| < 1, \quad 0 < |\xi| \leq \pi,$$

or

$$|z(\xi)| = 1, \quad |\xi| \leq \pi.$$

For the same reasons as in the continuous and semidiscrete problems, we realize that we only need to discuss the scalar problem without lower-order terms. The same results for a system of equations with lower-order terms will follow accordingly.

3.2. **Kreiss' condition and semigroup stability.** Consider the following fully discrete scheme :

$$\begin{cases} u_j^{n+1} = Q_0 u_j^n, & 0 \leq n \leq \infty, \\ u_j^0 = f_j, & \\ \mathscr{B} u_0^n = 0, & \end{cases} \qquad 1 \leq j \leq \infty, \tag{36}$$

where

$$Q_0 = I + kQ, \quad Q = \frac{1}{h} \sum_{j=-r}^{p} a_j E^j, \quad a_{-r} \neq 0, a_p \neq 0,$$

and a_j are polynomials in $\lambda = k/h$. The boundary operator $\mathscr{B}$ represents a set of functions

$$u_\mu = \sum_{j=1}^{q} \beta_{j\mu} u_{\mu+j}, \quad -r+1 \leq \mu \leq 0,$$

which make Q_0 well defined in the half-space.

According to Assumption 3.1, the solution of the corresponding Cauchy problem of (36) satisfies, for $0 \leq \lambda \leq \lambda_0$,

$$(u^n, u^n)_{-\infty,\infty} \leq e^{2\eta_0 nk}(u^0, u^0)_{-\infty,\infty}.$$

Similarly to the semidiscrete problem, we define

$$\langle u_j, u_j \rangle = \sum_{n=1}^{\infty} e^{-2\eta_0 nk} |u_j^n|^2 k.$$

There exists an analogue of Theorem 2.2 for fully discrete schemes:

Theorem 3.2. *There exists a special boundary operator $\mathscr{B}_0$ such that the solution of* (36) *satisfies*

$$\langle u_\mu, u_\mu \rangle \leq c_\mu (u^0, u^0), \quad -r+1 \leq \mu \leq 1. \tag{37}$$

where $c_\mu \geq 0$ depends on μ only.

Proof. In addition to the injection operator I, we define a projection operator $P: l^2(-\infty, \infty) \to l^2(-\infty, \infty)$ by

$$(Pu)_j = \begin{cases} u_j, & j = 1, 2, \dots, \\ 0, & j \leq 0. \end{cases}$$

We write the boundary terms as a vector $u_b^n \in l^2(-\infty, \infty)$:

$$(u_b^n)_j = \begin{cases} u_j^n, & -r+1 \leq j \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The energy can be estimated as follows:

$$\begin{aligned} (u^{n+1}, u^{n+1}) &= (Q_0 u^n, Q_0 u^n) \\ &= (PQ_0(Iu^n + u_b^n), PQ_0(Iu^n + u_b^n))_{-\infty,\infty} \\ &= (PQ_0 I u^n, PQ_0 I u^n)_{-\infty,\infty} + 2\operatorname{Re}(PQ_0 I u^n, PQ_0 u_b^n)_{-\infty,\infty} \\ &\quad + (PQ_0 u_b^n, PQ_0 u_b^n)_{-\infty,\infty} \\ &\leq e^{2\eta_0 k}(u^n, u^n) + 2\operatorname{Re}(Q_0 I u^n, Q_0 u_b^n)_{1,r} + (Q_0 u_b^n, Q_0 u_b^n)_{1,r}. \end{aligned}$$

Using the relation

$$(Q_0 I u^n)_j = (u^{n+1} - Q_0 u_b^n)_j, \quad 1 \leq j \leq r,$$

we obtain a simpler inequality,

$$\begin{aligned} (u^{n+1}, u^{n+1}) &\leq e^{2\eta_0 k}(u^n, u^n) + 2\operatorname{Re}(u^{n+1}, Q_0 u_b^n)_{1,r} \\ &= e^{2\eta_0 k}(u^n, u^n) + 2k\operatorname{Re}(u^{n+1}, Q u_b^n)_{1,r}. \end{aligned} \tag{38}$$

We now choose the following special boundary conditions:

$$\begin{aligned} u_\mu^n &= 0, \qquad -r+2 \leq \mu \leq 0, \\ u_{-r+1}^n &= -a_{-r}^{-1} u_1^{n+1}. \end{aligned} \tag{39}$$

Then, (38) is turned into

$$(u^{n+1}, u^{n+1}) + 2|a_{-r} u_{-r+1}^n|^2 k \leq e^{2\eta_0 k}(u^n, u^n). \tag{40}$$

Note that (39) implies

$$u_{-r+1}^{n} = -\frac{u_1^n + \lambda \sum_{i=0}^{p} a_i u_{i+1}^n}{(1+\lambda)a_{-r}}.$$

Multiply both sides of (40) by $e^{-2\eta_0(n+1)k}$ and sum up the inequality for n from 1 to $+\infty$ to obtain

$$2a_{-r}^2 \langle u_{-r+1}, u_{-r+1}\rangle = \langle u_1, u_1\rangle \le (u^0, u^0).$$

Naturally,

$$\langle u_\mu, u_\mu\rangle = 0, \quad -r+2 \le \mu \le 0.$$

Thus, we get the desired estimates for $\langle u_\mu, u_\mu\rangle$, $\mu = -r+1, \dots, 0$. We affiliate the set of boundary conditions (39) with the operator $\mathscr{B}_0$. □

Remark. There is more than one way to choose the special boundary operator. The above boundary operator has the advantage that

$$\langle (Q_0 u_b^n)_j, (Q_0 u_b^n)_j\rangle, \quad j = 1, \dots, r,$$

are bounded independently of λ. This fact is used in a forthcoming paper to discuss the estimates when $\lambda \to 0$.

The boundary operator $\mathscr{B}_0$ gives us the estimate of u_1 and allows us to proceed to the estimates of u_j for $j \ge 2$. Consider the following problem:

$$\begin{cases} u_j^{n+1} = Q_0 u_j^n, & 0 \le n \le \infty, \\ u_j^0 = f_j, & \\ u_\nu^n = u_\nu^n, & -r+2 \le \nu \le 1. \end{cases} \qquad j = 2, 3, \dots,$$

We split u_j^n, $j \ge 2$, into

$$u_j^n = v_j^n + (u_j^n - v_j^n), \qquad j = 2, 3, \dots, \tag{41}$$

where v_j^n, $j \ge 2$, is the solution of

$$\begin{cases} v_j^{n+1} = Q_0 v_j^n, & 0 \le n \le \infty, \\ v_j^0 = f_j, & \\ \mathscr{B}_0 v_1^n = 0, & -r+2 \le \nu \le 1, \end{cases} \qquad j = 2, 3, \dots,$$

and $w_j^n := u_j^n - v_j^n$, $j \ge 2$, satisfies

$$\begin{cases} w_j^{n+1} = Q_0 w_j^n, & \\ w_j^0 = 0, & j = 2, 3, \dots, \\ w_\mu^n = u_\mu^n - v_\mu^n, & -r+2 \le \mu \le 1. \end{cases}$$

From Theorem 3.2, we have

$$\langle v_\mu, v_\mu\rangle \le c_\mu(u^0, u^0), \quad -r+2 \le \mu \le 2.$$

The estimates of $\langle w_\mu, w_\mu\rangle$, $\mu \ge 2$, will then rely on the GKS theory. Denoting the Laplace transforms of the step functions u_j, v_j, and w_j by $\hat{u}_j$, $\hat{v}_j$, and

$\hat{w}_j$, respectively, we have

$$\begin{cases} z\hat{w}_j = Q_0\hat{w}_j, & j = 2, 3, \dots, \\ \hat{w}_\mu = \hat{u}_\mu - \hat{v}_\mu, & -r+2 \le \mu \le 1, \end{cases} \tag{42}$$

where $z = e^{sk}$. By the same arguments as those used in showing Lemma 2.3, we can prove

Lemma 3.2. *The eigenvalue problem corresponding to* (42) *satisfies the Kreiss condition.*

Thus, by Lemma 3.1, we get

$$\langle w_j, w_j\rangle \le \text{const} \sum_{-r+2}^{1} (\langle u_\mu, u_\mu\rangle + \langle v_\mu, v_\mu\rangle) \le \text{const}(u^0, u^0), \quad j \ge 2,$$

and therefore

$$\langle u_2, u_2\rangle \le \text{const}(\langle v_2, v_2\rangle + \langle w_2, w_2\rangle) \le \text{const}(u^0, u^0).$$

Using the same approach as that for the estimate of u_2, we can get estimates for u_j, $j \ge 3$. Hence we obtain

Lemma 3.3. *For equation* (36) *with the special boundary operator* $\mathscr{B}_0$, *we have*

$$\langle u_j, u_j\rangle \le c_j(u^0, u^0), \quad j \ge 1, \tag{43}$$

where c_j *depends on* j *only.*

For (36) with general boundary conditions, we split its solution into

$$u_j^n = v_j^n + (u_j^n - v_j^n), \quad j = 1, 2, \dots,$$

where v_j^n, $j \ge 1$, is the solution of (36) with the special boundary condition $\mathscr{B}_0 v_0^n = 0$, and $w_j^n := u_j^n - v_j^n$, $j \ge 1$, satisfies

$$\begin{cases} w_j^{n+1} = Q_0 w_j^n, & 0 \le n \le \infty, \\ w_j^0 = 0, \\ \mathscr{B} w_0^n = -\mathscr{B} v_0^n. \end{cases} \quad j = 1, 2, \dots,$$

Knowing that

$$\langle v_j, v_j\rangle \le c_j(u^0, u^0), \quad j \ge 1, \tag{44}$$

we now prove

Theorem 3.3. *Under Assumption* 3.1, *the IBV problem* (36) *is stable in the semigroup sense if the Kreiss condition is satisfied.*

Proof. From (38), we have

$$\begin{aligned} &e^{-2\eta_0(n+1)k}(w^{n+1}, w^{n+1}) \\ &\quad\le e^{-2\eta_0 nk}(w^n, w^n) + 2e^{-2\eta_0(n+1)k}(w^{n+1}, Q_0 w_b^n)_{1,r} \\ &\quad\le e^{-2\eta_0 nk}(w^n, w^n) + \text{const}\, e^{-2\eta_0(n+1)k} \left(\sum_{j=1}^{p} (|w_j^{n+1}|^2 + \sum_{j=-r+1}^{0} |w_j^n|^2 \right) k, \end{aligned}$$

where $p = \max\{q, r\}$. If the Kreiss condition holds, then by Lemma 3.1 and Lemma 3.3, we have

$$\langle w_j, w_j\rangle \le \text{const}\sum_{i=1}^{p}\langle v_i, v_i\rangle \le \text{const}(u^0, u^0), \qquad 1 \le j \le p.$$

Hence,

$$(w^n, w^n) \le \text{const}\, e^{2\eta_0 nk}(u^0, u^0).$$

The bound for u^n comes from the triangle inequality

$$\|u^n\| \le \|v^n\| + \|w^n\| \le \text{const}\, e^{2\eta_0 nk}\|u^0\|.$$

This completes the proof. □

We finish this paper by stating the corresponding result for a system of equations.

Theorem 3.4. *Under Assumption* 3.1, *the IBV problem* (5)–(7) *is stable in the semigroup sense if the Kreiss condition is satisfied.*

Remark. The results we have for one-dimensional hyperbolic equations can be generalized to multidimensional hyperbolic equations with symmetric coefficient matrices. It can be seen that, with Michelson's version [12] of the GKS theory for the discretized problems in multidimensional space, one can derive the corresponding results along approaches similar to those for one-dimensional problems.

Bibliography

1. M. S. Agronovich, *Theorem of matrices depending on parameters and its application to hyperbolic systems*, Functional Anal. Appl. **6** (1972), 85–93.
2. M. Goldberg and E. Tadmor, *Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems.* II, Math. Comp. **36** (1981), 603–626.
3. B. Gustafsson, *Numerical boundary conditions*, Lectures in Appl. Math., vol. 22, 1985, pp. 279–308.
4. B. Gustafsson, H.-O. Kreiss, and A. Sundström, *Stability theory of difference approximations for mixed initial boundary value problems.* II, Math. Comp. **26** (1972), 649–686.
5. R. Hersch, *Mixed problems in several variables.* J. Math. Mech. **12** (1963), 317–334.
6. H.-O. Kreiss, *Difference approximations for the initial-boundary value problem for hyperbolic differential equations*, Numerical Solutions of Nonlinear Differential Equations (Proc., D. Greenspan, ed.), Wiley, New York, 1966, pp. 141–166.
7. ———, *Stability theory for difference approximations of mixed initial boundary value problems.* I, Math. Comp. **22** (1968), 703–714.
8. ———, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math. **23** (1970), 277–298.
9. ———, *Problems with different time scales for partial differential equations*, Comm. Pure Appl. Math. **33** (1980), 399–439.
10. H.-O. Kreiss, *Difference approximation for initial boundary value problems.* I, preprint.
11. H.-O. Kreiss and L. Wu, *On the stability definitions of difference approximations for the initial boundary value problems*, Appl. Numer. Math. **12** (1993), 213–227.
12. D. Michelson, *Stability theory of difference approximations for multidimensional initial boundary value problems*, Math. Comp. **40** (1983), 1–46.

13. S. Osher, *Systems of difference equations with general homogeneous boundary conditions*, Trans. Amer. Math. Soc. **137** (1969), 171–201.
14. J. Rauch, *L_2 is a continuable condition for Kreiss' mixed problems*, Comm. Pure Appl. Math. **25** (1972), 62–69.
15. J. Strikwerda, *Initial boundary value problems for the methods of lines*, J. Comput. Phys. **34** (1980), 94–107.

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG

E-mail address: malwu@masu4.ust.hk